Harnack inequality of a doubly nonlinear parabolic equations with variable exponents

H. El Bahja

December 10, 2017

Abstract

In this paper, due to a modified version of Bombieri Lemma, we present a sampler proof of the Harnack inequality for positive solutions of a doubly nonlinear parabolic equations satisfy $p(x)$-growth conditions.

keywords:p(x)-Laplacian, Harnack inequality, Generalized Sobolev space.

1 Introduction

Let Ω be an open domain in the Euclidean space $\mathbb{R}^N$ and T be an open interval on the real line. For $(x,t) \in \Omega \times T$ we consider positive weak solutions of the following parabolic equation

$$\frac{d}{dt}\left(u^{p(x)-1}\right) - div\left(|\nabla u|^{p(x)-2}\nabla u\right) = 0, \tag{1.1}$$

where $p(x)$ is a measurable function in Ω and

$$1 < p^- = \inf_{\Omega} p(x) \le p(x) \le p^+ = \sup_{\Omega} p(x) < \infty.$$

Equation of type (1.1) are classified as doubly nonlinear parabolic equations and their study dates back to 1970s (see [10]) and recently [2,3]. Equation (1.1) can be regarded as a generalized form of two sorts of nonlinear diffusion equations: porous medium equation ($p(x) = 2$) and doubly nonlinear parabolic p-Laplace equation ($p(x) = p$), and moreover, it also appears in some model of non-Newtonian fluid dynamics. This field has also encouraged the developments of the theory of nonlinear evolution equations (see,e.g., [3,16]).

Harnack estimates for parabolic equations play an important role in discussing regularity of solutions to the corresponding equations. Moser in [12,13] derived Harnack inequalities for second order parabolic equations with bounded and measurable coefficients and Trudinger in [15] proved that the solutions of (1.1) actually satisfy classical Harnack inequalities for ($p(x) = p$). Later, the result in [13,15] was independently generalized to more general parabolic equations by the authors in [3,5].

The most difficult aspect of the proof of Harnack's inequality of (1.1) is the adaptation to the parabolic case of the well-known Lemma of F. John and L. Nirenberg , which concerns the exponential decay of the distribution function of a function with bounded mean oscillation. Consequently, our main objective is to give a relatively simple and transparent proof for Harnack's inequality using the approach in [10,13,14]. In particular, the parabolic John-Nirenberg lemma is replaced with a lemma due to Bombieri in [4] with $p(x)$-growth conditions.

Three key points in proving Theorem 2.2 lie in constructing some cylinders, selecting some testing functions and determining mean value inequalities for supersolutions and subsolutions of (1.1). In Section 2, we describe our assumptions and results, more precisely we give a modified version of Bombieri's Lemma which is the keystone of this work . Next, in section 3 and 4 we show how localized Sobolev inequalities imply certain $L^{p(x)}$ mean value inequalities, for subsolutions and supersolutions of (1.1). Finally, in section 5 we give an Harnack inequality for positive supersolutions and positive solutions of (1.1).

2 Preliminary results

To show that our proof is based on a general principle we consider the case where the Lebesgue measure is replaced with a more general Borel measure μ such that the measure is nontrivial in the sense that the measure of every nonempty open set is strictly positive and measure of every bounded set is finite. Moreover, let μ satisfy the following two properties :

1. there exits a constant $D_0 \geq 1$ such that
$$\mu\left(B(x,2R)\right) \leq D_0 \mu\left(B(x,R)\right),$$
where $B(x,R)$ denotes the open ball with center $x \in \mathbb{R}^N$ and radius $R > 0$.

2. there exit constants $P_0 > 0$, $s > 1$ and $\delta \geq 1$ such that
$$\fint_{B(x,R)} \left|u - u_{B(x,R)}\right| \, d\mu \leq P_0 R \left(\fint_{B(x,\delta R)} |\nabla u|^s \, d\mu \right)^{\frac{1}{s}}.$$

Here we use the notation
$$u_{B(x,R)} = \fint_{B(x,R)} u \, d\mu = \frac{1}{\mu\left(B(x,R)\right)} \int_{B(x,R)} u d\mu.$$

Definition 2.1. *$u(x,t) \in L^{p(x)}\left(0, T, W^{1,p(x)}(\Omega)\right)$ is a solution of the problem (1.1) if*
$$\int_0^T \int_\Omega \left(|\nabla u|^{p(x)} \nabla u . \nabla \varphi - u^{p(x)-1} \frac{d\varphi}{dt} \right) d\mu dt = 0, \tag{2.1}$$
for all $\varphi \in C_0^\infty\left((0,T) \times \Omega\right)$. Further, u a supersolution (resp subsolution) if (2.1) is nonnegative (resp nonpositive) for all $\varphi \in C_0^\infty\left((0,T) \times \Omega\right)$ and $\varphi \geq 0$.

The existence of weak solutions, in the sense of (2.1), was proved in [2]. The main result of the paper is given below.

Theorem 2.2. *Fix $0 < \delta < 1$ and $0 < R \leq \infty$. Then, there exits a constant $C = C(p^+, p^-, \varrho, D_0, P_0, \delta)$ such that, for $x \in \mathbb{R}^N$, $s \in \mathbb{R}$, $0 < r < R$ and any positive function $u \geq \varrho > 0$ satisfying (2.1) in $U = B(x,r) \times (s - r^{p^-}, s + r^{p^-})$, we have*
$$ess \sup_{U_\delta^-} u \leq C \; ess \inf_{U_\delta^+} u, \tag{2.2}$$
where
$$U_\delta^+ = B(x, \delta r) \times (s + \frac{1}{2} r^{p^-} - \frac{1}{2}(\delta r)^{p^-}, s + \frac{1}{2} r^{p^-} + \frac{1}{2}(\delta r)^{p^-}),$$
$$U_\delta^- = B(x, \delta r) \times (s - \frac{1}{2} r^{p^-} - \frac{1}{2}(\delta r)^{p^-}, s - \frac{1}{2} r^{p^-} + \frac{1}{2}(\delta r)^{p^-}).$$

Now we will present an elementary but subtle lemma due to Bombieri and Giusti [4] which simplifies considerably Moser's original proof of the Harnack inequality. It replaces the use of the well-known John-Nirenberg inequality. Therefore, consider a collection of measurable subsets U_δ, $0 < \delta \leq 1$, of a fixed measure space endowed with a measure $\overline{\mu}$, such that $U_{\delta'} \subset U_\delta$ if $\delta' \leq \delta$.

In what follows, we denote by the natural product measure on $\mathbb{R} \times \mathbb{R}^N$:

$$d\overline{\mu} = dt \times d\mu.$$

Lemma 2.3. *Fix $0 < \xi < 1$. Let θ, A and γ be positive constants and $0 < \alpha_0 \leq \infty$. Let f be a positive measurable function on $U_1 = U$ which satisfies*

$$\left(\fint_{U_{\delta'}} f^{\alpha_0} d\overline{\mu} \right)^{\frac{1}{\alpha_0}} \leq \left(\frac{A}{(\delta - \delta')^\theta} \left(\fint_{U_\delta} f^{\alpha} d\overline{\mu} + 1 \right) \right)^{\frac{1}{\alpha'}},$$

for all $\delta, \delta', \alpha, \alpha'$ such that $0 < \xi \leq \delta' < \delta \leq 1$, $0 < \alpha < \alpha_0$ and $\alpha' = k\alpha$, $\forall k > 0$. Assume further that f satisfies

$$\overline{\mu} \left(\log f > \lambda \right) \leq \frac{A \overline{\mu}(U)}{\lambda^\gamma},$$

for all $\lambda > 0$. Then

$$\left(\fint_{U_\xi} f^{\alpha_0} d\overline{\mu} \right)^{\frac{1}{\alpha_0}} \leq C,$$

where C depends on $\xi, \theta, \alpha, \alpha'$ and A.

Proof. Assume without loss of generality that $\overline{\mu}(U) = 1$ and set

$$\psi = \psi(\delta) = \log \left(\fint_{U_\delta} f^{\alpha_0} d\overline{\mu} \right)^{\frac{1}{\alpha_0}}.$$

Decomposing U_δ into the sets where $\log f \leq \frac{\psi}{2}$ and $\log f > \frac{\psi}{2}$, we get

$$\fint_{U_\delta} f^\alpha d\overline{\mu} \leq \exp\left(\psi \frac{\alpha}{2} \right) + \exp(\psi\alpha) \left(\frac{A}{\left(\frac{\psi}{2} \right)^\gamma} \right)^{\dfrac{\alpha_0 - \alpha}{\alpha_0}}, \tag{2.3}$$

here, we have used successively the Holder inequality and the second hypothesis of our Lemma. Next, we want to choose α so that the two terms in the right-hand side of (2.3) are equal and $0 < \alpha \leq \alpha_0$. This is possible if

$$0 < \log \left(\frac{\psi^\gamma}{A 2^\gamma} \right) \leq \alpha_0 \psi,$$

and this last inequality is certainly satisfied when

$$\psi \geq A_1, \tag{2.4}$$

where A_1 depends on A, γ, α_0. If we choose

$$\alpha = \frac{2}{3} \psi^{-1} \log \left(\frac{\psi^\gamma}{A 2^\gamma} \right),$$

then $0 < \alpha < \alpha_0$ and we have

$$\fint_{U_\delta} f^\alpha d\overline{\mu} \leq 2\exp\left(\frac{\alpha\psi}{2}\right). \tag{2.5}$$

This and the first hypothesis of our Lemma yield

$$\begin{aligned}\psi(\delta') \leq& \frac{1}{\alpha'}\log\left(\frac{A}{(\delta-\delta')^\theta}\left(\fint_{U_\delta} f^\alpha d\overline{\mu}+1\right)\right)\\ \leq& \frac{\psi}{2k}\left(3\frac{\log\left(\frac{3A}{(\delta-\delta')^\theta}\right)}{\log\left(\frac{\psi^\gamma}{A2^\gamma}\right)}+1\right).\end{aligned} \tag{2.6}$$

Here we have two cases, the first case if $k \geq 1$. On the one hand, we assume that

$$\frac{\psi^\gamma}{A2^\gamma} \geq \left(\frac{3A}{(\delta-\delta')^\theta}\right)^6, \tag{2.7}$$

then,

$$\psi(\delta') \leq \frac{3}{4k}\psi(\delta).$$

On the other hand, if one of the hypotheses (2.4) and (2.7) made on ψ is not satisfied , we have

$$\psi(\delta') \leq \psi(\delta) \leq A_1 + \frac{A_2}{(\delta-\delta')^{\frac{6\theta}{\gamma}}}, \tag{2.8}$$

where A_2 depends only on A and γ.
In all cases, we obtain

$$\psi(\delta') \leq \frac{3}{4k}\psi(\delta) + A_2(\delta-\delta')^{\frac{-6\theta}{\gamma}}. \tag{2.9}$$

For any sequence

$$0 < \xi = \delta_0 < \delta_1 < ... < \delta_i \leq 1,$$

an iteration of (2.9) yields

$$\psi(\delta_0) \leq \left(\frac{3}{4k}\right)^i \psi(\delta_i) + A_2\sum_{j=0}^{i-1}\left(\frac{3}{4k}\right)^j(\delta_{j+1}-\delta_j)^{\frac{-6\theta}{\gamma}} \tag{2.10}$$

and, when i tends to infinity

$$\psi(\delta) \leq A_2\sum_{j=0}^{\infty}\left(\frac{3}{4k}\right)^j(\delta_{j+1}-\delta_j)^{\frac{-6\theta}{\gamma}}. \tag{2.11}$$

The desired bound follows with $\delta_j = 1-(1+j)^{-1}(1-\xi)$. For the second case we take $0 < k < 1$, we suppose that

$$\frac{\psi^\gamma}{A2^\gamma} \geq \left(\frac{3A}{(\delta-\delta')^\theta}\right)^{3k'}, \tag{2.12}$$

where k' is a positive constant, then

$$\psi(\delta') \leq \frac{\psi}{k}\left(\frac{1+k'}{2k'}\right).$$

Therefore by the same method of the first case, for any sequence

$$0 < \xi = \delta_0 < \delta_1 < \ldots < \delta_i \leq 1,$$

and by iteration, we get

$$\psi(\delta_0) \leq \left(\frac{1}{k}\left(\frac{1+k'}{2k'}\right)\right)^i \psi(\delta_i) + A_2 \sum_{j=0}^{i-1} \left(\frac{1}{k}\left(\frac{1+k'}{2k'}\right)\right)^j (\delta_{j+1} - \delta_j)^{\frac{-k'\theta}{\gamma}}. \tag{2.13}$$

Now, we want to choose k' in a way that

$$0 < \frac{1}{k}\left(\frac{1+k'}{2k'}\right) < 1,$$

that is possible if we take k' big enough such that

$$0 < \frac{1+k'}{2k'} = \frac{1}{2} + \varepsilon < k < 1,$$

with ε infinitely small.
Then, by letting i tends to infinity and taking $\delta_j = 1-(1+j)^{-1}(1-\delta)$, we get the desired result. □

3 Mean value inequalities for subsolutions

The aim of this section is to show a local boundedness for $u \geq \varrho > 0$ a subsolution of (2.1) in $\Omega \times (t_1, t_2)$ by using Moser's iteration technique. In order to begin this argument we will construct the following Caccioppoli inequality.

Lemma 3.1. *Let $\varepsilon > 0$ and $u \geq \varrho > 0$ be a subsolution of (2.1). Then, there exits a positive constant $C = C(p^+, p^-, \varepsilon)$ such that*

$$\begin{aligned} \underset{t_1<t<t_2}{ess\sup} \int_\Omega u^{p^-+\varepsilon-1}\varphi^\beta d\mu + \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p^-} u^{\varepsilon-1}\varphi^\beta d\bar{\mu} \\ \leq C\Bigg(\int_{t_1}^{t_2}\int_\Omega u^{p(x)+\varepsilon-1}|\nabla\varphi|^{p(x)} d\bar{\mu} + \int_{t_1}^{t_2}\int_\Omega u^{p(x)+\varepsilon-1}\left|\frac{\partial}{\partial t}\varphi\right| d\bar{\mu} \\ + \underset{t_1<t<t_2}{ess\sup}\int_\Omega \varphi^\beta d\mu + \int_{t_1}^{t_2}\int_\Omega u^{\varepsilon-1}\varphi^\beta d\bar{\mu}\Bigg), \end{aligned} \tag{3.1}$$

for every $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ with $\varphi \geq 0$ and $\beta > p^+$.

Proof. For any nonnegative function $\phi \in C_0^\infty(\Omega \times (t_1, t_2))$ we have

$$\int_{t_1}^{t_2}\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\phi - u^{p(x)-1}\frac{\partial\phi}{\partial t}\right) d\bar{\mu} \leq 0. \tag{3.2}$$

Then, by taking $\phi = u^{\varepsilon}\varphi^{\beta}$ we get the following weak formulation for all $t_1 < \tau_1 < \tau_2 < t_2$

$$
\begin{aligned}
&-\int_{\tau_1}^{\tau_2}\int_{\Omega} \varepsilon u^{\varepsilon+p(x)-2}\varphi^{\beta} d\bar{\mu} - \int_{\tau_1}^{\tau_2}\int_{\Omega} u^{\varepsilon+p(x)-1}\frac{\partial \varphi^{\beta}}{\partial t} d\bar{\mu} + \left[\int_{\Omega} u^{\varepsilon+p(x)-1}\varphi^{\beta} d\mu\right]_{t=\tau_1}^{\tau_2} \\
&\quad + \int_{\tau_1}^{\tau_2}\int_{\Omega} \varepsilon \left|\nabla u\right|^{p(x)} u^{\varepsilon-1}\varphi^{\beta} d\bar{\mu} + \int_{\tau_1}^{\tau_2}\int_{\Omega} \left|\nabla u\right|^{p(x)-2}\nabla u \ u^{\varepsilon}\nabla\varphi^{\beta} d\bar{\mu} \leq 0.
\end{aligned}
\tag{3.3}
$$

Since,

$$
\frac{-\varepsilon}{p(x)+\varepsilon-1}\frac{\partial}{\partial t} u^{p(x)+\varepsilon-1} = -\varepsilon u^{p(x)+\varepsilon-2}\frac{\partial u}{\partial t},
$$

we integrate by parts and get

$$
\begin{aligned}
&\varepsilon \int_{\tau_1}^{\tau_2}\int_{\Omega} \left|\nabla u\right|^{p(x)} u^{\varepsilon-1}\varphi^{\beta} d\bar{\mu} + \left(1 - \frac{\varepsilon}{p^{-}+\varepsilon-1}\right)\left[\int_{\Omega} u^{\varepsilon+p(x)-1}\varphi^{\beta} d\mu\right]_{t=\tau_1}^{\tau_2} \\
&\leq \left(\frac{\varepsilon}{p^{-}+\varepsilon-1} - 1\right)\int_{\tau_1}^{\tau_2}\int_{\Omega} u^{\varepsilon+p(x)-1}\frac{\partial \varphi^{\beta}}{\partial t} d\bar{\mu} + \int_{\tau_1}^{\tau_2}\int_{\Omega} \left|\nabla u\right|^{p(x)-1} u^{\varepsilon}\left|\nabla\varphi^{\beta}\right| d\bar{\mu}.
\end{aligned}
\tag{3.4}
$$

By Young's inequality, there exists a positive constant ν such that

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{\Omega} \left|\nabla u\right|^{p(x)-1} u^{\varepsilon}\left|\nabla\varphi^{\beta}\right| d\bar{\mu} \\
&= \int_{\tau_1}^{\tau_2}\int_{\Omega}\left(\left|\nabla u\right| \varphi^{\frac{\beta}{p(x)}} u^{\frac{\varepsilon-1}{p(x)}}\right)^{p(x)-1}\left(\left|\nabla\varphi^{\beta}\right| \varphi^{\frac{-\beta(p(x)-1)}{p(x)}} u^{\varepsilon+\frac{(1-\varepsilon)(p(x)-1)}{p(x)}}\right) d\bar{\mu} \\
&\leq \nu \int_{\tau_1}^{\tau_2}\int_{\Omega} \left|\nabla u\right|^{p(x)} u^{\varepsilon-1}\varphi^{\beta} d\bar{\mu} + C(\nu)\int_{\tau_1}^{\tau_2}\int_{\Omega} u^{\varepsilon+p(x)-1}\left|\nabla\varphi^{\beta}\right|^{p(x)}.
\end{aligned}
\tag{3.5}
$$

Furthermore, by using (3.5), choosing $t_1 = \tau_1$ and letting $\tau_2 \longrightarrow \tau < t_2$ such that

$$
\frac{1}{2} \underset{t_1<t<t_2}{ess\sup} \int_{\Omega} u^{\varepsilon+p(x)-1}\varphi^{\beta} d\mu \leq \int_{\Omega} u^{\varepsilon+p(x)-1}(x,\tau)\varphi^{\beta}(x,\tau),
$$

then, (3.4) become

$$
\begin{aligned}
&\underset{t_1<t<t_2}{ess\sup} \int_{\Omega} u^{p(x)+\varepsilon-1}\varphi^{\beta} d\mu + \int_{t_1}^{t_2}\int_{\Omega} \left|\nabla u\right|^{p(x)} u^{\varepsilon-1}\varphi^{\beta} d\bar{\mu} \leq \\
&\quad C\left(\int_{t_1}^{t_2}\int_{\Omega} u^{p(x)+\varepsilon-1}\varphi^{\beta(1-p(x))}\left|\nabla\varphi^{\beta}\right|^{p(x)} d\bar{\mu} + \int_{t_1}^{t_2}\int_{\Omega} u^{p(x)+\varepsilon-1}\left|\frac{\partial}{\partial t}\varphi^{\beta}\right| d\bar{\mu}\right).
\end{aligned}
\tag{3.6}
$$

Next, we estimate the left terms of (3.6) by Young's inequality, then there exist positive constants ν_1 and ν_2 such that

$$
\int_{\Omega} u^{p^{-}+\varepsilon-1}\varphi^{\beta} d\mu \leq \nu_1 \int_{\Omega} u^{p(x)+\varepsilon-1}\varphi^{\beta} d\mu + C(\nu_1)\int_{\Omega} \varphi^{\beta} d\mu,
\tag{3.7}
$$

and,

$$
\int_{t_1}^{t_2}\int_{\Omega} \left|\nabla u\right|^{p^{-}} u^{\varepsilon-1}\varphi^{\beta} d\bar{\mu} \leq \nu_2 \int_{t_1}^{t_2}\int_{\Omega} \left|\nabla u\right|^{p(x)} u^{\varepsilon-1}\varphi^{\beta} d\bar{\mu} + C(\nu_2)\int_{t_1}^{t_2}\int_{\Omega} u^{\varepsilon-1}\varphi^{\beta} d\bar{\mu}.
\tag{3.8}
$$

Putting (3.7)-(3.8) into (3.6) leads to (3.1). □

Before stating the main result of this section, we introduce some further notations. Fix a parameter $\tau > 0$. Consider $x \in \Omega$, $r > 0$ and $s \in \mathbb{R}^N$. Consider also a parameter δ, $0 < \delta \leq 1$ and set

$$Q(\tau, s, r) = Q = B(x, r) \times (s - \tau r^{p^-}, s + \tau r^{p^-})$$

$$Q_\delta = B(x, \delta r) \times (s - \tau(\delta r)^{p^-}, s + \tau(\delta r)^{p^-}).$$

Theorem 3.2. *Let $u \geq \varrho > 0$ be subsolution of (2.1) in $\Omega \times (t_1, t_2)$. Then, there exist positive constants $C_1(\tau, p^+, p^-)$ and $C_2(\chi, p^+, p^-)$ where $\chi = 2 - \frac{p^-}{k}$ and $k > p^-$ such that, for any real s, any $0 < \delta^{'} < \delta \leq 1$, we have*

$$\underset{Q_{\delta'}}{ess\sup}\ u \leq \frac{C_1}{(\delta - \delta')^{C_2}} \left(\fint_{Q_\delta} u^\alpha d\bar{\mu} + 1 \right)^{\frac{1}{\alpha'}} \tag{3.9}$$

for all $\alpha, \alpha^{'} > 0$.

Proof. We will start by choosing cut-off functions $\varphi_j \in C_0^\infty(Q_j)$ such that

$$0 \leq \varphi_j \leq 1,\ \varphi_j = 1 \ in\ Q_{j+1},$$

By Holder's inequality in the spatial integral and by estimating the first factor by essential supremum we get

$$\begin{aligned} &\fint_{Q_{j+1}} u^{p^- + \varepsilon - 1 + (p^- + \varepsilon - 1)\frac{k - p^-}{k}} \\ &\leq C \left(\underset{T_j}{ess\sup} \fint_{B_j} u^{p^- + \varepsilon - 1} \varphi_j^\beta d\mu \right)^{\frac{k-p^-}{k}} \times \fint_{T_j} \left(\fint_{B_j} \left| u^{p^- + \varepsilon - 1} \varphi_j^\beta \right|^{\frac{k}{p^-}} d\mu \right)^{\frac{p^-}{k}} dt. \end{aligned} \tag{3.10}$$

Using the Sobolev embedding and Lemma 3.1 yields

$$\begin{aligned} &\fint_{Q_{j+1}} u^{p^- + \varepsilon - 1 + (p^- + \varepsilon - 1)\frac{k - p^-}{k}} d\bar{\mu} \\ &\leq C \left(\underset{T_j}{ess\sup} \fint_{B_j} u^{p^- + \varepsilon - 1} \varphi_j^\beta d\mu \right)^{\frac{k-p^-}{k}} \times \fint_{T_j} \fint_{B_j} \left| \nabla \left(u^{\frac{p^- + \varepsilon - 1}{p^-}} \varphi_j^{\frac{\beta}{p^-}} \right) \right|^{p^-} d\bar{\mu} \\ &\leq C \left(\underset{T_j}{ess\sup} \fint_{B_j} u^{p^- + \varepsilon - 1} \varphi_j^\beta d\mu + \fint_{Q_j} u^{\varepsilon - 1} |\nabla u|^{p^-} \varphi_j^\beta d\bar{\mu} + \fint_{Q_j} u^{p^- + \varepsilon - 1} |\nabla \varphi_j|^{p^-} d\bar{\mu} \right)^{2 - \frac{p^-}{k}} \\ &\leq C \Bigg(\fint_{Q_j} u^{p^- + \varepsilon - 1} |\nabla \varphi_j|^{p^-} d\bar{\mu} + \fint_{Q_j} u^{p(x) + \varepsilon - 1} |\nabla \varphi_j|^{p(x)} d\bar{\mu} + \fint_{Q_j} u^{p(x) + \varepsilon - 1} \left| \frac{\partial}{\partial t} \varphi_j \right| d\bar{\mu} \\ &\quad + \underset{T_j}{ess\sup} \fint_{B_j} \varphi_j^\beta d\mu + \fint_{Q_j} u^{\varepsilon - 1} \varphi_j^\beta d\bar{\mu} \Bigg)^{2 - \frac{p^-}{k}}, \end{aligned} \tag{3.11}$$

Now, by taking

$$|\nabla \varphi_j| \leq \frac{2^j C}{r(\delta - \delta')},\ |\nabla \varphi_j|^\xi = \max\left(|\nabla \varphi_j|^{p^+}, |\nabla \varphi_j|^{p^-} \right),\ \text{and}\ \left| \frac{\partial \varphi_j}{\partial t} \right| \leq \frac{C}{\tau} \left(\frac{2^j}{r(\delta - \delta')} \right)^\xi,$$

(3.11) become

$$\fint_{Q_{j+1}} u^{p^-+\varepsilon-1+(p^-+\varepsilon-1)\frac{k-p^-}{k}} d\bar{\mu} \leq C\left(\frac{2^{j\xi}}{(\delta-\delta')^{\xi}}\left(\fint_{Q_j} u^{p^++\varepsilon-1} d\bar{\mu}+1\right)\right)^{2-\frac{p^-}{k}} \tag{3.12}$$

Now we are ready to start Moser's iteration. Let $\chi = 2-\frac{p^-}{k}$ and $\varepsilon_0 = \alpha - p^+ + 1$ with $\alpha > p^+ - 1$. For $j \geq 1$ define recursively

$$\pi_{j+1} = p^+ + \varepsilon_{j+1} - 1 = p^- + \varepsilon_j - 1 + (p^- + \varepsilon_j - 1)\frac{k-p^-}{k}.$$

Solving for ε_j gives

$$\varepsilon_j = \frac{\chi^j}{\chi-1}\left(\chi(\alpha+p^--p^+)-\alpha\right) + \frac{1-p^+ + \chi\,(p^--1)}{1-\chi},$$

we should take α bigger enough to make sure that $\varepsilon_j > 0$, $\forall j \geq 1$. Moreover, denote

$$\psi_j = \fint_{Q_j} u^{\pi_j(x)} d\bar{\mu}$$

and hence by iteration

$$\psi_{j+1} \leq C\frac{2^{j\xi\chi}}{(\delta-\delta')^{\xi\chi}}(\psi_j+1)^{\chi} \leq \ldots \leq \frac{C^{\gamma^*}2^{\xi C_{prod}}2^{\gamma^*-1}}{(\delta-\delta')^{\xi\gamma_{prod}}}(\psi_0+1)^{\chi^{j+1}}, \tag{3.13}$$

where

$$\gamma^* = \frac{\chi^{j+1}-1}{\chi-1},\ C_{prod} = \sum_{k=0}^{j}\gamma^{k+1}(j-k) \text{ and } \gamma_{prod} = \frac{\chi^{j+2}-\chi}{\chi-1}.$$

Altogether we have

$$\psi_{j+1}^{\frac{1}{\pi_{j+1}}} \leq \frac{C^{\frac{\gamma^*}{\pi_{j+1}}}2^{\frac{\xi C_{prod}}{\pi_{j+1}}}2^{\frac{\gamma^*-1}{\pi_{j+1}}}}{(\delta-\delta')^{\frac{\xi\gamma_{prod}}{\pi_{j+1}}}}(\psi_0+1)^{\frac{\chi^{j+1}}{\pi_{j+1}}}. \tag{3.14}$$

It's easy to verify that $C^{\frac{\gamma^*}{\pi_{j+1}}}2^{\frac{\xi C_{prod}}{\pi_{j+1}}}2^{\frac{\gamma^*-1}{\pi_{j+1}}} < \infty\ \forall j \geq 1$, and that both $\dfrac{\gamma_{prod}}{\pi_{j+1}}$ and $\dfrac{\chi^{j+1}}{\pi_{j+1}}$ tend to $\dfrac{1}{\alpha-p^++p^-}$ as $j \longrightarrow \infty$. Next, we take $\alpha' = \alpha - p^+ + p^-$ and $Q_j = Q_{\delta_j}$, such that

$$\delta_j = \delta - (\delta-\delta')(1-2^{-j}).$$

Hence, by letting $j \longrightarrow \infty$ in (3.14), we get the desired result. □

4 Mean value inequalities for supersolutions

The main tool used in the preceding section is the mean value inequality for subsolutions stated in Theorem 2.3, supersolutions $u \geq \varrho > 0$ satisfy similar but different inequalities that are presented below. In order to begin this argument we will construct the following Caccioppoli inequality.

Lemma 4.1. *Let $\varepsilon > p^+ - 1$ and $u \geq \varrho > 0$ be a supersolution of (2.1). Then, there exits a positive constant $C = C(p^+, p^-, \varepsilon)$ such that*

$$
\begin{aligned}
&\underset{t_1<t<t_2}{ess\sup} \int_\Omega u^{p^- - \varepsilon - 1} \varphi^\beta d\mu + \int_{t_1}^{t_2} \int_\Omega |\nabla u|^{p^-} u^{-\varepsilon-1} \varphi^\beta d\bar{\mu} \\
&\qquad \leq C\Big(\int_{t_1}^{t_2} \int_\Omega u^{p(x)-\varepsilon-1} |\nabla \varphi|^{p(x)} d\bar{\mu} + \int_{t_1}^{t_2} \int_\Omega u^{p(x)-\varepsilon-1} \left| \frac{\partial}{\partial t} \varphi \right| d\bar{\mu} \\
&\qquad\qquad + \underset{t_1<t<t_2}{ess\sup} \int_\Omega \varphi^\beta d\mu + \int_{t_1}^{t_2} \int_\Omega u^{-\varepsilon-1} \varphi^\beta d\bar{\mu} \Big),
\end{aligned} \tag{4.1}
$$

for every $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ with $\varphi \geq 0$ and $\beta > p^+$.

Proof. For any nonnegative function $\phi \in C_0^\infty(\Omega \times (t_1, t_2))$ we have

$$
\int_{t_1}^{t_2} \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla \phi - u^{p(x)-1} \frac{\partial \phi}{\partial t} \right) d\bar{\mu} \geq 0. \tag{4.2}
$$

Then, by taking $\phi = u^{-\varepsilon} \varphi^\beta$ and using the same argument of Lemma 3.1 we get the desired result. □

Now, we need to show an estimation for the essential supremum of u^{-1}. We also going to use the same notation of Theorem 3.2. This, along with the Sobolev style embedding inequality will provide the basis of the Moser-type iteration

Theorem 4.2. *Let $u \geq \varrho > 0$ be a supersolution of (2.1) in $\Omega \times (t_1, t_2)$. Then, there exist positive constants $C_1(\tau, \varrho, p^+, p^-)$ and $C_2(\chi, p^+, p^-)$ where $\chi = 2 - \frac{p^-}{k}$ and $k > p^-$ such that, for any real s, any $0 < \delta' < \delta \leq 1$ and any $\alpha, \alpha' > 0$, we have*

$$
\underset{Q_{\delta'}}{ess\sup}\, u^{-1} \leq \frac{C_1}{(\delta - \delta')^{C_2}} \left(\fint_{Q_\delta} \left(u^{-1}\right)^\alpha d\bar{\mu} + 1 \right)^{\frac{1}{\alpha'}} . \tag{4.3}
$$

Proof. We give the proof assuming $\tau = 1$. Letting $\varepsilon > 2p^+$, and $v = u^{-1}$ in (4.1). This yields

$$
\begin{aligned}
&\underset{t_1<t<t_2}{ess\sup} \int_\Omega v^{\varepsilon+1-p(x)} \varphi^\beta d\mu + \int_{t_1}^{t_2} \int_\Omega |\nabla v|^{p(x)} v^{\varepsilon+1-2p(x)} \varphi^\beta d\bar{\mu} \\
&\qquad \leq C\Big(\int_{t_1}^{t_2} \int_\Omega v^{\varepsilon+1-p(x)} |\nabla \varphi|^{p(x)} d\bar{\mu} + \int_{t_1}^{t_2} \int_\Omega v^{\varepsilon+1-p(x)} \left| \frac{\partial}{\partial t} \varphi \right| d\bar{\mu} \Big),
\end{aligned} \tag{4.4}
$$

Young's inequality gives

$$
\begin{aligned}
&\underset{t_1<t<t_2}{ess\sup} \int_\Omega v^{\varepsilon+1-p^+} \varphi^\beta d\mu + \int_{t_1}^{t_2} \int_\Omega |\nabla v|^{p^-} v^{\varepsilon+1-2p(x)} \varphi^\beta d\bar{\mu} \\
&\qquad \leq C\Big(\int_{t_1}^{t_2} \int_\Omega v^{\varepsilon+1-p(x)} |\nabla \varphi|^{p(x)} d\bar{\mu} + \int_{t_1}^{t_2} \int_\Omega v^{\varepsilon+1-p(x)} \left| \frac{\partial}{\partial t} \varphi \right| d\bar{\mu} \\
&\qquad\qquad + \underset{t_1<t<t_2}{ess\sup} \int_\Omega \varphi^\beta d\mu + \int_{t_1}^{t_2} \int_\Omega v^{\varepsilon+1-2p(x)} \varphi^\beta d\bar{\mu} \Big).
\end{aligned} \tag{4.5}
$$

Next, let the choices of the test functions, it's derivatives and Q_j be the same as in the proof of Theorem 3.2 with the exception that $\delta_j = \delta - (\delta - \delta')(1 - 2^{-j})$. This, along with (4.5) and the

following inequality: for a given $\theta > 0$, there is a positive constant $C(\theta)$ depending only on θ such that

$$|\log(v)| \le C(\theta) + v^{\frac{\theta}{2}} + v^{\frac{-\theta}{2}}, \quad \forall v > 0, \tag{4.6}$$

give

$$\begin{aligned}
&\fint_{Q_{j+1}} v^{\varepsilon+1+p^- -2p(x)+(\varepsilon+1-p^+)\frac{k-p^-}{k}} \\
&\quad \le C\Bigg(\operatorname*{ess\,sup}_{T_j} \fint_{B_j} v^{\varepsilon+1-p^+}\varphi_j^{\beta} d\mu + \fint_{Q_j} |\nabla v|^{p^-} v^{\varepsilon+1-2p(x)}\varphi_j^{\beta} d\bar{\mu} \\
&\quad + \fint_{Q_j} v^{\varepsilon+1+p^- -2p(x)}|\nabla \varphi_j|^{p^-} d\bar{\mu} + \fint_{Q_j} |\log v|^{p^-} v^{\varepsilon+1+p^- -2p(x)}\varphi_j^{\beta} d\bar{\mu}\Bigg)^{\chi} \\
&\quad \le C\left(\frac{2^{j\xi}}{(\delta-\delta')^{\xi}} \left(\fint_{Q_j} v^{\varepsilon+1} d\bar{\mu} + 1\right)\right)^{\chi},
\end{aligned} \tag{4.7}$$

here we used the same idea in (3.12) and took $\theta = 2$. Now, using young's inequality on the left term of (4.7) give us

$$\fint_{Q_{j+1}} v^{\varepsilon+1+p^- -2p^+ +(\varepsilon+1-p^+)\frac{k-p^-}{k}} \le C\left(\frac{2^{j\xi}}{(\delta-\delta')^{\xi}} \left(\fint_{Q_j} v^{\varepsilon+1} d\bar{\mu} + 1\right)\right)^{\chi}. \tag{4.8}$$

From here, the same iteration used in (3.14) yields (4.3), such that $\alpha' = \alpha - p^+$ and α is bigger enough to make sure that $\varepsilon_j > 2p^+$. □

Under the same notation of Theorem 3.2, the next result is a reverse Holder inequality for positive supersolutions of (2.1).

Theorem 4.3. *Fix $\tau > 0$ and $0 < \alpha_0 < (p^- - 1)\chi$ where $\chi = 2 - \frac{p^-}{k}$ and $k > p^-$. Let $u \ge \varrho > 0$ be a supersolution of (2.1) in $\Omega \times (t_1, t_2)$. Then, there exist positive constants $C_1(\tau, p^+, p^-)$ and $C_2(\kappa, p^+, p^-)$ such that, for any real s, any $0 < \delta' < \delta \le 1$, any $0 < \alpha < \frac{\alpha_0}{\chi}$, we have*

$$\left(\fint_{Q_{\delta'}} u^{\alpha_0} d\bar{\mu} \right)^{\frac{1}{\alpha_0}} \le \frac{C_1}{(\delta-\delta')^{C_2}} \left(\fint_{Q_\delta} u^{\alpha} d\bar{\mu} + 1\right)^{\frac{1}{\alpha}}. \tag{4.9}$$

Proof. Let $\tau = 1$ and the choices of the test functions, it's derivatives and Q_j be the same as in the proof of the previous theorem with the exception that

$$\delta_j = \delta - (\delta - \delta')\frac{1-2^{-j}}{1-2^{-i}}\,.$$

Furthermore, let $\lambda = p^- - \varepsilon - 1$ and $0 < \varepsilon < p^- - 1$. Then, as in the proof of Theorem 3.2 we obtain from Sobolev's inequality and Lemma 4.1 that

$$\fint_{Q_{j+1}} u^{\lambda\chi} d\bar{\mu} \le C\left(\frac{2^{j\xi}}{(\delta-\delta')^{\xi}} \left(\fint_{Q_j} u^{p(x)-\varepsilon-1} d\bar{\mu} + 1\right)\right)^{\chi}. \tag{4.10}$$

Next, by Young's inequality we simplify the following

$$
\begin{aligned}
u^{p(x)-\varepsilon-1} &= u^{p^-}\left(\left(u^{p(x)-p^-}\right)^{\frac{p(x)-1}{p(x)}}\left(\left(u^{p(x)-p^-}\right)^{\frac{1}{p(x)}}.u^{-\varepsilon-1}\right)\right) \\
&\le C\ u^{p^-}\left(u^{p(x)-p^-}+u^{p(x)(-\varepsilon-1)}\right) \\
&\le C\ u^{p^-}\left(u^{p(x)-p^-}+u^{-\varepsilon-1}.u^{\varepsilon(1-p(x))+(1-p^-)}\right) \\
&\le C\left(u^{p(x)}+u^{p^- -\varepsilon-1}\right),
\end{aligned}
\tag{4.11}
$$

here we use the fact that u is a positive supersolution.
Therefore, using (4.10), (4.11) and the fact that $u \in L^{p(x)}(0,T,W^{1,p(x)}(\Omega))$ give

$$
\fint_{Q_{j+1}} u^{\lambda\chi} d\bar{\mu} \le C\left(\frac{2^{j\xi}}{(\delta-\delta')^{\xi}}\left(\fint_{Q_j} u^{\lambda} d\bar{\mu}+1\right)\right)^{\chi}. \tag{4.12}
$$

This is analogous to Theorem 3.2, but the iterative steps that will now be used to finish the proof of Theorem 4.3 are somewhat different from those used in the proof of Theorem 3.2.
Define $\alpha_i = \alpha_0\chi^{-i}$. Next, fix $i \ge 1$ and apply (4.12) with $\lambda = \alpha_i\chi^{j-1}$, $j = 1, 2, .., i$. Observe that $\alpha_i\chi^{j-1} \le p^- - 1$, for $j = 1.2, .., i$, as required for the validity of (4.12). Hence, for all $j = 1, 2, .., i$,

$$
\fint_{Q_i} u^{\alpha_i\chi^j} d\bar{\mu} \le C\left(\frac{2^{j\xi}}{(\delta-\delta')^{\xi}}\left(\fint_{Q_{i-1}} u^{\alpha_i\chi^{j-1}} d\bar{\mu}+1\right)\right)^{\chi}. \tag{4.13}
$$

This yields

$$
\left(\fint_{Q_i} u^{\alpha_0} d\bar{\mu}\right)^{\frac{1}{\alpha_0}} \le \left(\frac{C^{\gamma_1}2^{\xi\gamma_2}}{(\delta-\delta')^{\xi\gamma_3}}\left(\fint_{Q_0} u^{\alpha_i} d\bar{\mu}+1\right)\right)^{\frac{1}{\alpha_i}}, \tag{4.14}
$$

where $\gamma_1 = \frac{\chi^i-1}{\chi-1}$, $\gamma_2 = \sum_{j=0}^{i-1}(i-j)\chi^j$ and $\gamma_3 = \frac{\chi^{i+1}-\chi}{\chi-1}$. Obviously the constants γ_1, γ_2 and γ_3 are uniformly bounded on i. To obtain the desired inequality for any $\alpha \in (0, \frac{\alpha_0}{\chi})$, let $i \ge 2$ be the integer such that $\alpha_i \le \alpha \le \alpha_{i-1}$. Thus, by Holder we get

$$
\left(\fint_{Q_i} u^{\alpha_0} d\bar{\mu}\right)^{\frac{1}{\alpha_0}} \le \left(\frac{C_1}{(\delta-\delta')^{C_2}}\left(\fint_{Q_0} u^{\alpha} d\bar{\mu}+1\right)\right)^{\frac{1}{\alpha}}. \tag{4.15}
$$

Hence, by the previous definition of δ_i such that $Q_i = Q_{\delta_i}$, we get the desired result. $\square$

5 Harnack inequalities

5.1 An inequality for $\log u$

Now, following the shames in [1,13], we are going to prove that the second condition of Lemma 2.3 holds, which allows us to have all the necessary tools to proof the main result of these article. Therefore, consider a parameter δ, $0 < \delta \le 1$ and set

$$
Q(\tau, s, r) = Q = B(x, r) \times (s - \tau r^{p^-}, s + \tau r^{p^-}),
$$

$$
Q_\delta^+ = B(x, \delta r) \times (s, s + \tau(\delta r)^{p^-})
$$

and

$$
Q_\delta^- = B(x, \delta r) \times (s - \tau(\delta r)^{p^-}, s).
$$

Theorem 5.1. *Fix $0 < R \leq \infty$. Fix $\tau > 0$ and $\delta \in (0,1)$. For any real s, any r with $0 < r < R$ and any positive supersolutions $u \geq \varrho > 0$ in Q and for $\beta > p^+$ let $Y = \frac{1}{N}\int_{B(x,r)} \log u(x,s)\varphi^\beta d\mu$. Then, there exit constants $C = C(p^+, p^-, D_0, \delta, \tau)$ and $C' = C'(p^+, p^-, D_0, \delta, \tau)$ such that, for all $\lambda > 0$ we have*

$$\overline{\mu}\left(\{(x,t) \in Q_\delta^- \ : \ \log u(x,t) > \lambda + Y + C'\}\right) \leq \frac{C}{\lambda^{p^-} - 1}\overline{\mu}\left(Q_\delta^-\right),$$

and

$$\overline{\mu}\left(\{(x,t) \in Q_\delta^+ \ : \ \log u(x,t) < -\lambda + Y - C'\}\right) \leq \frac{C}{\lambda^{p^-} - 1}\overline{\mu}\left(Q_\delta^+\right).$$

N and φ will be defined later in the proof.

To proof this theorem, we shall begin with the following lemma.

Lemma 5.2. *Assume that $\nabla p \in L^\infty(\Omega \times (t_1,t_2))$. Let $u \geq \varrho > 0$ be a supersolution in $\Omega \times (t_1,t_2)$. There exit constants $C_1 = C(p^+, p^-, \nabla p)$ and $C_2 = C(p^+, p^-, \varrho)$ such that*

$$\begin{aligned} &C_1 \int_{t_1}^{t_2}\int_\Omega |\nabla(\log u)|^{p^-}\ \varphi^\beta d\bar{\mu} + \underset{t_1<t<t_2}{ess\sup}\int_\Omega \log u\ \varphi^\beta d\mu \\ &\qquad \leq C_2\bigg(\int_{t_1}^{t_2}\int_\Omega |\log u|\left|\frac{d}{dt}\varphi^\beta\right| d\bar{\mu} + \int_{t_1}^{t_2}\int_\Omega \varphi^\beta d\bar{\mu} + \int_{t_1}^{t_2}\int_\Omega \varphi^{\beta-p(x)}\left|\nabla\varphi\right|^{p(x)} d\bar{\mu}\bigg), \end{aligned} \tag{5.1}$$

for every $\varphi \in C_0^\infty(\Omega \times (t_1,t_2))$ with $\varphi \geq 0$.

Proof. Formally we choose the test function $\phi = u^{1-p(x)}\varphi^\beta$ so that

$$\nabla\phi = (1-p(x))u^{-p(x)}\nabla u\ \varphi^\beta + \nabla p \log\left(\frac{1}{u}\right)u^{1-p(x)}\varphi^\beta + u^{1-p(x)}\nabla(\varphi^\beta),$$

and

$$\frac{d}{dt}\phi = (1-p(x))u^{-p(x)}\frac{du}{dt}\varphi^\beta + u^{1-p(x)}\frac{d}{dt}\varphi^\beta.$$

where $\varphi \in C_0^\infty(\Omega \times (t_1,t_2))$ with $\varphi \geq 0$. Let $t_1 < \tau_1 < \tau_2 < t_2$. We integrate by parts and get

$$\begin{aligned} 0 &\leq -\int_{t_1}^{t_2}\int_\Omega u^{p(x)-1}\frac{d\phi}{dt}\ d\bar{\mu} + \left[\int_\Omega u^{p(x)-1}\phi\ d\mu\right]_{\tau_1}^{\tau_2} + \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\phi\ d\bar{\mu} \\ &= \int_{t_1}^{t_2}\int_\Omega (p(x)-1)\left(\frac{d}{dt}\log u - |\nabla \log u|^{p(x)}\right)\varphi^\beta d\bar{\mu} \\ &+ \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p(x)-2}\nabla u\ u^{1-p(x)}\nabla\left(\varphi^\beta\right)d\bar{\mu} \\ &+ \int_{t_1}^{t_2}\int_\Omega \nabla p\,|\nabla u|^{p(x)-2}\nabla u\ u^{1-p(x)}\log\left(\frac{1}{u}\right)\varphi^\beta d\bar{\mu} \\ &\leq C\bigg(\left[\int_\Omega \log u\varphi^\beta d\mu\right]_{\tau_1}^{\tau_2} - \int_{t_1}^{t_2}\int_\Omega |\nabla \log u|^{p(x)}\varphi^\beta d\bar{\mu} \\ &- \int_{t_1}^{t_2}\int_\Omega \log u\frac{d}{dt}\varphi^\beta d\bar{\mu} + \int_{t_1}^{t_2}\int_\Omega |\nabla \log u|^{p(x)-1}\varphi^\beta d\bar{\mu} + \int_{t_1}^{t_2}\int_\Omega |\nabla \log u|^{p(x)-1}\nabla\left(\varphi^\beta\right)d\bar{\mu}\bigg). \end{aligned} \tag{5.2}$$

where C is a positive constant depends on $p^+,\ p^-\ ,\|\nabla p\|_\infty$ and ϱ. We set $v=\log u$, then (5.2) become

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega |\nabla v|^{p(x)}\varphi^\beta d\bar\mu-\left[\int_\Omega v\varphi^\beta\ d\mu\right]_{\tau_1}^{\tau_2}\\ &\le C\Big(\int_{t_1}^{t_2}\int_\Omega |v|\left|\frac{d}{dt}\varphi^\beta\right|d\bar\mu+\int_{t_1}^{t_2}\int_\Omega|\nabla v|^{p(x)-1}\varphi^\beta d\bar\mu+\int_{t_1}^{t_2}\int_\Omega\beta\,|\nabla v|^{p(x)-1}\varphi^{\beta-1}\nabla\varphi\ d\bar\mu\Big).\end{aligned}\tag{5.3}$$

Now, by Young's inequality, there exits a positive constant ε such that

$$\int_{t_1}^{t_2}\int_\Omega|\nabla v|^{p(x)-1}\varphi^\beta d\bar\mu\le\varepsilon\int_{t_1}^{t_2}\int_\Omega|\nabla v|^{p(x)}\varphi^\beta d\bar\mu+C(\varepsilon)\int_{t_1}^{t_2}\int_\Omega\varphi^\beta d\bar\mu,\tag{5.4}$$

$$\int_{t_1}^{t_2}\int_\Omega|\nabla v|^{p(x)-1}\varphi^{\beta-1}\nabla\varphi\ d\bar\mu\le\varepsilon\int_{t_1}^{t_2}\int_\Omega|\nabla v|^{p(x)}\varphi^\beta d\bar\mu+C(\varepsilon)\int_{t_1}^{t_2}\int_\Omega\varphi^{\beta-p(x)}|\nabla\varphi|^{p(x)}\ d\bar\mu,\tag{5.5}$$

and,

$$\int_{t_1}^{t_2}\int_\Omega|\nabla v|^{p^-}\varphi^\beta d\bar\mu\le\varepsilon\int_{t_1}^{t_2}\int_\Omega|\nabla v|^{p(x)}\varphi^\beta d\bar\mu+C(\varepsilon)\int_{t_1}^{t_2}\int_\Omega\varphi^\beta d\bar\mu.\tag{5.6}$$

Consequently, we have

$$\begin{aligned}&C_1\int_{t_1}^{t_2}\int_\Omega|\nabla v|^{p^-}\varphi^\beta d\bar\mu-\left[\int_\Omega v\varphi^\beta\ d\mu\right]_{\tau_1}^{\tau_2}\\ &\le C_2\Big(\int_{t_1}^{t_2}\int_\Omega|v|\left|\frac{d}{dt}\varphi^\beta\right|d\bar\mu+\int_{t_1}^{t_2}\int_\Omega\varphi^\beta d\bar\mu+\int_{t_1}^{t_2}\int_\Omega\varphi^{\beta-p(x)}|\nabla\varphi|^{p(x)}\ d\bar\mu\Big).\end{aligned}\tag{5.7}$$

Hence, choosing $\tau_2=t_2$ and letting $\tau_1\longrightarrow\tau>t_1$ such that

$$\frac{1}{2}\underset{t_1<t<t_2}{ess\sup}\int_\Omega v\varphi^\beta d\mu\le\int_\Omega v(x,\tau)\varphi^\beta(x,\tau),$$

yield (5.1) $\square$

Now we are going to prove Theorem 5.1.

Proof. Let us assume $\tau=1$. We set $v(x,t)=\log u(x,t)-Y$ and let

$$\varphi(z,t)=\varphi(z)=\left(1-\frac{2|z-x|}{(1+\delta)r}\right)_+,$$

where $(z,t)\in B(x,r)\times(s-(\delta r)^{p^-},s+(\delta r)^{p^-})$. Then from (5.1) we get

$$\begin{aligned}&C_1\int_{t_1}^{t_2}\int_{B(x,r)}|\nabla v|^{p^-}\varphi^\beta d\bar\mu-\left[\int_{B(x,r)}v\varphi^\beta\ d\mu\right]_{t_1}^{t_2}\\ &\le C_2\Big(\int_{t_1}^{t_2}\int_{B(x,r)}\varphi^\beta d\bar\mu+\int_{t_1}^{t_2}\int_{B(x,r)}\varphi^{\beta-p(x)}\left(1+|\nabla\varphi|^{p^+}\right)\ d\bar\mu\Big),\end{aligned}\tag{5.8}$$

where $s-(\delta r)^{p^-}\le t_1\le t_2\le s+(\delta r)^{p^-}$.
Now, let

$$N=\int_{B(x,r)}\varphi^\beta(x)d\mu(x),$$

then

$$\left(\frac{1-\delta}{1+\delta}\right)^{\beta} \leq N \leq \mu(B(x,r)). \tag{5.9}$$

Also, let

$$M(t) = \frac{1}{N}\int_{B(x,r)} v(x,t)\varphi^{\beta}(x)d\mu(x).$$

Therefore, the weighted Poincaré inequality of Theorem 5.3.4 of [14] with the weight $\phi_x = \varphi^{\beta}$ yields

$$\int_{B(x,r)} |v - M(t)|^{p^-} \varphi^{\beta} d\mu \leq C r^{p^-} \int_{B(x,r)} |\nabla v|^{p^-} \varphi^{\beta} d\mu, \tag{5.10}$$

this with (5.8) give

$$\begin{aligned} C_1 \int_{B(x,r)} |\nabla v|^{p^-} \varphi^{\beta} d\mu &\geq \frac{1}{C r^{p^-}} \int_{B(x,r)} |v - M(t)|^{p^-} \varphi^{\beta} d\mu \\ &\geq \frac{(1-\delta)^{\beta}}{C r^{p^-}} \int_{B(x,\delta r)} |v - M(t)|^{p^-} d\mu. \end{aligned} \tag{5.11}$$

Using (5.7), (5.10) and the fact that

$$N \geq \int_{B(x,\frac{r}{4})} \varphi^{\beta} d\mu \geq 2^{-\beta}\mu(B(x,\frac{r}{4})) \geq 2^{-\beta} D_0^{-2}\mu(B(x,r)), \tag{5.12}$$

allow us to have

$$\frac{1}{C N r^{p^-}} \int_{t_1}^{t_2} \int_{B(x,r)} |v - M(t)|^{p^-} d\bar{\mu} + M(t_1) - M(t_2) \leq \frac{C'(t_2 - t_1)}{\tau(\delta r)^{p^+}}. \tag{5.13}$$

Rewrite this inequality as

$$\frac{1}{C N r^{p^-}} \int_{t_1}^{t_2} \int_{B(x,r)} \left|\overline{v} - \overline{M}(t)\right|^{p^-} d\bar{\mu} + \overline{M}(t_1) - \overline{M}(t_2) \leq 0, \tag{5.14}$$

where $\overline{v}(x,t) = v(x,t) + C' \dfrac{t-s}{\tau(\delta r)^{p^+}}$ and $\overline{M}(x,t) = M(t) + C' \dfrac{t-s}{\tau(\delta r)^{p^+}}$. Since $\overline{M}$ is a monotonic function it is differentiable almost everywhere. As a consequence we have

$$\frac{1}{C N r^{p^-}} \int_{t_1}^{t_2} \int_{B(x,r)} \left|\overline{v} - \overline{M}(t)\right|^{p^-} d\bar{\mu} + \frac{dt}{t}\overline{M}(t) \leq 0. \tag{5.15}$$

Now, for every t, $t_1 < t < t_2$, set

$$\Omega_t^-(\lambda)\{(x,t) \in Q_\delta^- \ : \ \overline{v}(x,t) > \lambda\},$$

and

$$\Omega_t^+(\lambda)\{(x,t) \in Q_\delta^+ \ : \ \overline{v}(x,t) < -\lambda\}.$$

Then, since $\overline{M}(t) \leq \overline{M}(s) = 0$ as $s > t > t - (\delta r)^{p^-}$ we have

$$\int_{B(x,r)} \left|\overline{v} - \overline{M}(t)\right|^{p^-} d\mu \geq \left(\lambda - \overline{M}(t)\right)^{p^-} \bar{\mu}\left(\Omega_t^-(\lambda)\right) \geq \lambda^{p^-} \bar{\mu}\left(\Omega_t^-(\lambda)\right). \tag{5.16}$$

Using this in (5.14), we obtain

$$-\frac{\frac{dt}{t}\overline{M}(t)}{\left(\lambda-\overline{M}(t)\right)^{p^-}}+C\frac{\bar{\mu}\left(\Omega_t^-(\lambda)\right)}{Nr^{p^-}}\leq 0, \tag{5.17}$$

for almost every $s>t>s-(sr)^{p^-}$.
Integrating from $s-(sr)^{p^-}$ to s and obtain

$$\frac{\bar{\mu}\left(\Omega_t^-(\lambda)\right)}{Nr^{p^-}}\leq C\left[\left(\lambda-\overline{M}(t)\right)^{-(p^--1)}\right]_{s-(sr)^{p^-}}^{s}\leq\frac{C}{\lambda^{p^--1}}. \tag{5.18}$$

This implies

$$\overline{\mu}\left(\{(x,t)\in Q_\delta^- \ : \ \log u(x,t)>\lambda+Y+C'\}\right)\leq\frac{C}{\lambda^{p^--1}}\overline{\mu}\left(Q_\delta^-\right).$$

Working with $\Omega_t^+(\lambda)$ instead of $\Omega_t^-(\lambda)$, we obtain the second inequality by a similar argument. $\square$

5.2 Harnack inequality for positive supersolutions

The following theorem states that positive supersolutions satisfy a weak form of Harnack inequality. For any fixed $\tau>0$, $\delta\in(0,1)$ and $s,r>0$ define

$$U=Q=B(x,r)\times(s-r^{p^-},s+r^{p^-}),$$

$$U_\delta^+=B(x,\delta r)\times(s+\frac{1}{2}r^{p^-}-\frac{1}{2}(\delta r)^{p^-},s+\frac{1}{2}r^{p^-}+\frac{1}{2}(\delta r)^{p^-})$$

and

$$U_\delta^-=B(x,\delta r)\times(s-\frac{1}{2}r^{p^-}-\frac{1}{2}(\delta r)^{p^-},s-\frac{1}{2}r^{p^-}+\frac{1}{2}(\delta r)^{p^-}).$$

By the same notation of the pervious sections, we have the following

Theorem 5.3. *Fix $\rho_0\in(0,(p^--1)\chi)$. Let $u\geq\varrho>0$ be a supersolution in U. Then, there exits a constant $C=C\left(p^-,p^+,D_0,\xi\right)$, such that for $0<\xi<1$ we have*

$$\left(\fint_{U_\xi^-}u^{\rho_0}d\bar{\mu}\right)^{\frac{1}{\rho_0}}\leq ess\inf_{U_\xi^+} u. \tag{5.19}$$

Proof. Let Y and C' the constants given by Theorem 5.1. Set $v_2=u\exp(-Y-C')$, then Theorem 4.3 gives

$$\left(\fint_{U_{\delta'}^-}v_2^{\rho_0}d\bar{\mu}\right)^{\frac{1}{\rho_0}}\leq\frac{C_1}{(\delta-\delta')^{C_2}}\left(\fint_{U_{\delta'}^-}v_2^{\rho}d\bar{\mu}+1\right)^{\frac{1}{\rho}}, \tag{5.20}$$

for all $0<\xi<\delta'<\delta<\frac{1+\xi}{2}$ and $0<\rho<\rho_0\chi^{-1}$. By Theorem 5.1 we get

$$\overline{\mu}\left(\left\{(x,t)\in U_{\frac{1+\xi}{2}}^- \ : \ \log v_2(x,t)>Y\right\}\right)\leq\frac{C}{\lambda^{p^-}-1}\overline{\mu}\left(U_{\frac{1+\xi}{2}}^-\right).$$

Thus, we can apply Lemma 2.3 to conclude that

$$\left(\fint_{U_\xi^-}v_2^{\rho_0}d\bar{\mu}\right)^{\frac{1}{\rho_0}}\leq C. \tag{5.21}$$

Set now $v_1 = u^{-1}\exp(Y - C^{'})$. Then, by theorem 4.2 we have

$$\underset{U^+_{\delta'}}{ess\sup}\ v_1 \leq \frac{C_1}{(\delta-\delta')^{C_2}}\left(\fint_{U^+_{\delta}} v_1^{\rho} d\bar{\mu} + 1\right)^{\frac{1}{\rho'}}, \tag{5.22}$$

for all $0 < \xi < \delta' < \delta < \frac{1+\xi}{2}$ and $\rho, \rho' > 0$. By Theorem 5.1 we also have

$$\overline{\mu}\left(\left\{(x,t) \in U^+_{\frac{1+\xi}{2}} \ : \ \log v_1(x,t) > Y\right\}\right) \leq \frac{C}{\lambda p^{-} - 1}\overline{\mu}\left(U^+_{\frac{1+\xi}{2}}\right).$$

Consequently Lemma 2.3 gives

$$\underset{U^+_{\xi}}{ess\sup}\ v_1 \leq C. \tag{5.23}$$

Multiplying (5.20), (5.22) together, we obtain the desired result. □

Theorem 2.2 follows immediately from Theorems 3.2 and 5.3 with $\xi = \frac{1+\delta}{2}$.

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Hamid El Bahja: Laboratory of Analysis and Applications-URAC/03, Department of Mathematics, P.O. Box 1014, Faculty of Sciences, Mohammed V University, Rabat, Morocco.
E-mail address: hamidsm88@gmail.com.